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Villarceau foliated solitons in the two-dimensional O(3) nonlinear sigma model

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Abstract

We obtain the complete space of solutions, in the two-dimensional O(3) nonlinear sigma model, which are foliated by Villarceau circles. In particular, we prove the existence of solitons that admit a foliation by nonparallel circles. This contrasts with the Plateau context, where solitons foliated by nonparallel circles cannot exist. These solitons carry topological charges that are holographically computed via the Gauss–Bonnet formula.

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1. Introduction

The differential geometry of surfaces in Euclidean three-space has, even nowadays, plenty of exciting problems. Many of them are variational ones associated with functionals of the type

$$\mathcal{B}(M) = \int_M F(\mathrm{d}N)\,\mathrm{d}A,$$

acting on a certain class, $\mathcal{C}$, of surfaces, M, in $\mathbb{R}^3$ with Gauss map N, where F is a given function and $\mathrm{d}A$ is the element of area, on M, of the metric induced from the Euclidean one. Let us mention a few of them.

(1) In the class of surfaces in $\mathbb{R}^3$ with the same boundary, say a finite set, Γ, of regular closed curves, the following functional is considered:

$$\mathcal{P}(M) = \int_M \mathrm{d}A.$$

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The associated variational problem is the Plateau one with boundary Γ, $(\mathcal{P}, \Gamma)$. Its critical points give soap films or minimal surfaces (the mean curvature vanishes identically) bounded by Γ (see for example [12,22] and references therein).

(2) Let $\mathcal{C}$ be the class of closed (compact and boundary free) surfaces in $\mathbb{R}^3$ with a certain genus. In this space, there is considered the variational problem associated with the so-called Willmore functional, which measures the total squared mean curvature or the conformal total energy

$$\mathcal{W}(M) = \int_M H^2 \mathrm{d}A,$$

where $H = \frac{1}{2}\mathrm{trace}(\mathrm{d}N)$ is the mean curvature function of M. Now, the critical points are named Willmore surfaces (see for example [29,30] and references therein).

(3) In the class of surfaces in $\mathbb{R}^3$ with the same boundary and the same Gauss map along the common boundary, there is considered the functional

$$\mathcal{S}(M) = \int_M \|\mathrm{d}N\|^2 \mathrm{d}A,$$

which measures the total energy of the Gauss map (see for example [3,4] and references therein). Certainly, the functional $\mathcal{S}$ could be regarded as acting on the class of closed surfaces with a given genus. In this case, bearing in mind the Gauss–Bonnet theorem and the classical expression $|\mathrm{d}N|^2 = 4H^2 - 2G$ (where H is the mean curvature function and G is the Gaussian curvature), one has

$$\mathcal{S}(M) = 4\mathcal{W}(M) - 4\pi\chi(M),$$

where $\chi(M)$ is the Euler number of M. Since fluctuations do not change the topology, in the class of closed surfaces with a given genus, the Willmore problem is equivalent to that associated with the functional $\mathcal{S}$.

These variational problems, and others, are interesting not only in differential geometry but also because of their applications to a wide variety of nonlinear phenomena in physics and mathematics. For instance, a suitable coupling multiple of functional $\mathcal{P}$ gives the classical Nambu–Goto string action, which presents serious difficulties for quantization. To overcome these troubles, it is necessary to introduce in the string action the extrinsic geometry of the worldsheets. This was first done by Polyakov [25] and Kleinert [16], by considering the following bosonic string action on worldsheets M,

$$\mu_1\mathcal{P}(M) + \mu_2\mathcal{W}(M),$$

$\mu_1, \mu_2 \in \mathbb{R}$ being suitable coupling constants.

On the other hand, the Willmore functional is the main part of the elastic energy action, which is used in the study of biological membranes, resilient metal plates, interfaces between polymers and so on. In fact, around 1810, S. Germain proposed a simple geometric model for describing the elastic energy of a surface, given by the Lagrangian whose density is an even, symmetric function of the principal curvatures (the eigenvalues of $\mathrm{d}N$). Of course, the simplest choice is to assume that the Lagrangian density is quadratic, and so, it must involve both the squared mean curvature and the Gaussian one. Once again, if fluctuations do not change the topology of the membranes, the Gauss–Bonnet formula implies that the Hooke law for elastic energy is

$$\mathbf{E}(M) = a\mathcal{P}(M) + b\mathcal{W}(M),$$

where $a, b \in \mathbb{R}$ are constants related to the stretching energy and the bending energy, respectively. Thus, one obtains the popular Canham–Helfrich model for elastic surfaces [9,15].

Finally, the variational problem governed by the Lagrangian $\mathcal{S}$, that measures the energy of the Gauss maps, is mainly used in the two-dimensional O(3) nonlinear sigma model ($\mathbf{NSM}_2$). This is ubiquitous in physics (see for example [10,28] and references therein) where it is used in a wide range of fields, from condensed matter physics (see for example [8,20]) to high energy physics (see for instance [1,2]). The $\mathbf{NSM}_2$ plays an important role in string theories where the model description is applicable. Moreover, it has its own interest in differential geometry since, for example, it naturally leads to the appearance of partially integrable almost product structures [1,2].

The Plateau problem, $(\mathcal{C}, \Gamma)$, when Γ is made up of a pair of parallel circles (circles in two parallel planes) in $\mathbb{R}^3$ is classical in the theory of minimal surfaces, whose origin might be located in a posthumous manuscript of Riemann

[27]. In that paper, Riemann found out that the moduli space of all the minimal surfaces, M, with $\partial M = \Gamma$ foliated by parallel circles (i.e. contained in parallel planes), consists of the catenoid joint a one-parameter family of minimal surfaces, nowadays known as Riemann minimal surfaces. Later, Enneper [14] showed the non-existence of minimal surfaces foliated by nonparallel circles. Consequently, we know the complete moduli space of minimal surfaces that are foliated by circles.

In [7], the first author showed an algorithm for obtaining, explicitly, the moduli space of solitons in the $\mathbf{NSM_2}$ which are invariant under a rotational group of symmetries in $\mathbb{R}^3$. In particular, he obtained a wide family of solitons that are foliated by parallel circles. Therefore, it seems natural to state the following problem:

Problem 1. Are there any solitons in the $\mathbf{NSM_2}$ foliated by nonparallel circles?

In this paper, an affirmative answer to this problem is given by exhibiting the moduli space of solitons in the $\mathbf{NSM_2}$ that are foliated by Villarceau circles. The main steps in the construction are displayed in the following list:

(1) We show that the $\mathbf{NSM_2}$ is an invariant under conformal transformations of the Euclidean metric.
(2) We construct two conformal submersions from a part of $\mathbb{R}^3$ into the once-punctured round two-sphere with radius $\frac{1}{2}$.
(3) The fibres of the above-mentioned conformal submersions are Villarceau circles. For each submersion, there exists a one-parameter group of conformal transformations such that the orbits under its action are the fibres of the submersion.
(4) Thanks to the principle of symmetric criticality, it is shown that the family of those solutions in $\mathbf{NSM_2}$ foliated by Villarceau circles can be constructed by lifting, through the above-mentioned submersions, a certain class of clamped elasticae in the once-punctured two-sphere with radius $\frac{1}{2}$.

These examples of solitons in $\mathbf{NSM_2}$ foliated by nonparallel circles nicely contrast with the non-existence of such solutions in the Plateau context (see the above-mentioned result of Enneper). This could be expected after the wide class of solutions in the $\mathbf{NSM_2}$ which are invariant under a rotational group of symmetries [7], in contrast with the Plateau sigma model, where the catenoid is the only solution admitting such a class of symmetry.

The solitons of $\mathbf{NSM_2}$ carry topological charges which, from the Gauss–Bonnet formula, only depend on the boundary conditions. Finally, in the last section, we compute the curvature function of a Villarceau circle in a soliton. By evaluating the corresponding total curvature, we compute the topological charge that these solitons carry. It should be noticed that although that curvature function never vanishes identically, in other words no Villarceau circle is a geodesic of a soliton, the topological charge of a soliton could be zero.

2. The conformal invariance of the $\mathbf{NSM_2}$

The elementary fields in the $\mathbf{NSM}_2$ are $\mathbb{R}^3$-valued unit vector fields on surfaces, M, which coincide along the boundary, ∂M, if $\partial M \neq \emptyset$. An interesting approach for studying this model, in connection with the differential geometry of surfaces in the three-dimensional Euclidean space, was considered in [21]. In this context, one identifies the unit normal vector field, or, more correctly, the Gauss map of a surface in $\mathbb{R}^3$, with the dynamical variable of the $\mathbf{NSM}_2$. Needless to say, we are using the standard metric $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^3$. To be precise, let us consider the first-order boundary conditions (Γ, N_o), where

(1) $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a finite set of regular closed curves in $\mathbb{R}^3$ with $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$.
(2) N_o is a unit normal vector field along Γ such that $\langle N_o(p), \Gamma'(p)\rangle = 0, \forall p \in \Gamma$ where $\Gamma'(p) = \gamma_j'(p)$ if $p \in \gamma_j$.

In this setting, we have a vector field, ν, along Γ determined by $\Gamma'(p) \wedge \nu(p) = N_o(p), \forall p \in \Gamma$.

Let M be a differentiable surface with boundary $\partial M = c_1 \cup c_2 \cdots \cup c_n$. We denote by $I_\Gamma(M, \mathbb{R}^3)$ the space of immersions, $\phi : M \to \mathbb{R}^3$, that satisfy the following boundary conditions

(1) $\phi(\partial M) = \Gamma$, or $\phi(c_j) = \gamma_j$, $1 \leq j \leq n$, and
(2) $\mathrm{d}\phi_q(T_q M)$ is orthogonal to $N_o(\phi(q)), \forall q \in \partial M$.

For any $\phi \in I_\Gamma(M, \mathbb{R}^3)$, we have its Gauss map, $N_\phi : M \to \mathbb{S}^2$. Therefore, $\mathrm{d}N_\phi$ denotes the shape operator of ϕ. Now, the field configuration of the $\mathbf{NSM}_2$ can be identified with $I_\Gamma(M, \mathbb{R}^3)$ and the Lagrangian that governs the dynamics of the model, $\mathcal{S} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}$, measures the total energy of the Gauss maps, that is,

$$\mathcal{S}(\phi) = \int_M \|\mathrm{d}N_\phi\|^2 \mathrm{d}A_\phi,$$

where $\mathrm{d}A_\phi$ denotes the element of area of $(M, \phi^*\langle\cdot,\cdot\rangle)$. Roughly speaking, if we identify each immersion $\phi \in I_\Gamma(M, \mathbb{R}^3)$ with its graph $\phi(M)$, viewed as a surface with boundary in $\mathbb{R}^3$, then we propose the study of the Lagrangian $\mathcal{S}$ in the class of surfaces with the same boundary and with the same Gauss map along the common boundary.

If H_ϕ denotes the mean curvature function of $\phi \in I_\Gamma(M, \mathbb{R}^3)$ and we put $G_\phi = \det(\mathrm{d}N_\phi)$ to name the Gaussian curvature of $(M, \phi^*\langle\cdot,\cdot\rangle)$, the following relation is classical

$$\|\mathrm{d}N_\phi\|^2 = 4H_\phi^2 - 2G_\phi.$$

Certainly, the case $\partial M = \emptyset$ can be regarded as a particular one, and we put $I_\Gamma(M, \mathbb{R}^3) = I(M, \mathbb{R}^3)$. In this case, some known solutions to the $\mathbf{NSM}_2$ can be obtained from the theory of surfaces with constant mean curvature. For example, the solitons discovered by Belavin and Polyakov [8] correspond to those surfaces whose Gauss maps are conformal (round spheres and minimal surfaces). Also, the solutions given by Purkait and Ray [26] are induced by the family of constant mean curvature helicoids studied by Do Carmo and Dajzer [13]. Anyway, when M is assumed to be compact and free of boundary, then one can combine the above-mentioned relation with the Gauss–Bonnet theorem to obtain

Theorem 2. *Let M be a compact and boundary free surface, then $\phi \in I(M, \mathbb{R}^3)$ is a soliton of the* $\mathbf{NSM}_2$ *if and only if (M, ϕ) is a Willmore surface, that is, ϕ is a critical point of the action $\mathcal{W} : I(M, \mathbb{R}^3) \to \mathbb{R}$, given by*

$$\mathcal{W}(\phi) = \int_M H_\phi^2 \mathrm{d}A_\phi.$$

This result can be used to generate wide families of compact solitons of the free $\mathbf{NSM}_2$ (combine, for example, with the classes of Willmore surfaces obtained in [5,6,18,19,24]).

The amazing fact is that the $\mathbf{NSM}_2$ for any (Γ, N_o) and the Willmore problem with these boundary conditions [29] are also equivalent, producing a result similar to that obtained in Theorem 2 for the closed (compact and boundary free) case. To prove this claim, we observe that the $\mathbf{NSM}_2$ action can be written as

$$\mathcal{S}(\phi) = \int_M \|\mathrm{d}N_\phi\|^2 \mathrm{d}A_\phi = 4\int_M H_\phi^2 \mathrm{d}A_\phi - 2\int_M G_\phi \mathrm{d}A_\phi.$$

However, according to the Gauss–Bonnet formula, we have

$$\int_M G_\phi \mathrm{d}A_\phi + \sum_{i=1}^n \int_{\gamma_i} k_i^\phi \mathrm{d}s = 2\pi\chi(M),$$

where k_i^ϕ stands for the curvature function of γ_i in $\phi(M)$ endowed with the $\langle\cdot,\cdot\rangle$-induced metric, s is the arc length parameter and $\chi(M)$ is the Euler characteristic of M. Then,

$$\mathcal{S}(\phi) = \int_M \|\mathrm{d}N_\phi\|^2 \mathrm{d}A_\phi = 4\int_M H_\phi^2 \mathrm{d}A_\phi + 2\sum_{i=1}^n \int_{\gamma_i} k_i^\phi \mathrm{d}s - 4\pi\chi(M),$$

and so, the variational problem associated with $\mathcal{S}$ is equivalent with that defined via the action $\mathcal{M} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}$ given by

$$\mathcal{M}(\phi) = 4\int_M H_\phi^2 \mathrm{d}A_\phi + 2\sum_{i=1}^n \int_{\gamma_i} k_i^\phi \mathrm{d}s.$$

On the other hand, the functional $\mathcal{L} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}$ given by

$$\mathcal{L}(\phi) = \int_{\phi(\partial M)} k^\phi \mathrm{d}s = \sum_{i=1}^n \int_{\gamma_i} k_i^\phi \mathrm{d}s,$$

is constant on the whole $I_\Gamma(M,\mathbb{R}^3)$. Indeed, the stated boundary conditions imply that the curvature, k^ϕ, of $\phi(\partial M) = \Gamma$ in $\phi(M)$, endowed with the $\langle\cdot,\cdot\rangle$-induced metric, does not depend on $\phi \in I_\Gamma(M,\mathbb{R}^3)$ because k^ϕ comes from the projection of the boundary acceleration, Γ'', on $\mathrm{d}\phi_p(T_pM)$, which is the tangent plane of each $\phi(M)$, since all the immersions have the same Gauss map along the common boundary Γ. This allows us, from now on, to write k as k^ϕ. In particular, we have

Theorem 3. *Let M be a compact surface. Then $\phi \in I_\Gamma(M,\mathbb{R}^3)$ is a soliton of the* $\mathbf{NSM}_2$ *with boundary conditions (Γ, N_o) if, and only if, (M,ϕ) is a Willmore surface, that is, ϕ is a critical point of the action $\mathcal{W}: I_\Gamma(M,\mathbb{R}^3) \to \mathbb{R}$, given by*

$$\mathcal{W}(\phi) = \int_M H_\phi^2 \mathrm{d}A_\phi + \int_\Gamma k\,\mathrm{d}s,$$

with the same boundary conditions.

As a consequence of Theorems 2 and 3, we obtain the conformal invariance of the $\mathbf{NSM}_2$.

3. Villarceau circles, Clifford parallelism in the three-sphere and more

Given $\mathbf{T}$ a revolution torus in $\mathbb{R}^3$, it is well known that $\mathbf{T}$ contains two families of circles, the parallels of latitude and the meridians. However, it is less well known that $\mathbf{T}$ contains other circles as well. These are sometimes named *Villarceau circles* (1848) and can be found by intersecting $\mathbf{T}$ with a bitangent plane. In fact, one can find two families, $\mathcal{F}_1 = \{\Upsilon(t)\}$ and $\mathcal{F}_2 = \{\Xi(t)\}$, of these exotic circles. Two circles of different families intersect in exactly two points while two circles of the same family not only do not intersect, but they are always linked.

Let us consider the unit three-sphere

$$\mathbb{S}^3 = \{\zeta = (z_1, z_2) \in \mathbb{C}^2 : |\zeta|^2 = |z_1|^2 + |z_2|^2 = 1\}.$$

Then, we have the usual action $\mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3$ defined by

$$(\mathrm{e}^{\mathrm{i}t}, \zeta) \mapsto \mathrm{e}^{\mathrm{i}t}\cdot\zeta = (\mathrm{e}^{\mathrm{i}t}z_1, \mathrm{e}^{\mathrm{i}t}z_2).$$

The orbits under this action are great circles (geodesics) of $\mathbb{S}^3$. If $\mathbf{C}$ and $\mathbf{C}'$ denote any two orbits, then we have

$$\mathrm{d}(\zeta, \mathbf{C}') = \mathrm{d}(\eta, \mathbf{C}') \quad \text{for any } \zeta, \eta \in \mathbf{C}.$$

Moreover, if $\zeta \in \mathbf{C}$ and $\zeta' \in \mathbf{C}'$ satisfy $\mathrm{d}(\zeta,\zeta') = \mathrm{d}(\mathbf{C},\mathbf{C}')$, then any great circle containing ζ and ζ' intersects orthogonally both $\mathbf{C}$ and $\mathbf{C}'$. This leads to the following definition. Two great circles, $\mathbf{C}$ and $\mathbf{C}'$, in $\mathbb{S}^3$ are Clifford parallel if $\mathrm{d}(\zeta, \mathbf{C}')$ does not depend on $\zeta \in \mathbf{C}$. If this is the case, then we write $\mathbf{C}\|\mathbf{C}'$.

Given a great circle, $\mathbf{C}$, in $\mathbb{S}^3$ and $\theta \in [0,\pi]$, we define

$$\mathbf{C}_\theta = \{\zeta \in \mathbb{S}^3 : \mathrm{d}(\zeta, \mathbf{C}) = \theta\}.$$

We put $\mathbf{C}^\perp$ to denote the great circle associated with the plane, through the origin, that is orthogonal to that corresponding to $\mathbf{C}$. It is obvious that $\mathbf{C}_0 = \mathbf{C}$, $\mathbf{C}_{\frac{\pi}{2}} = \mathbf{C}^\perp$, $\mathbf{C}_{\frac{\pi}{2}-\theta} = \mathbf{C}_\theta^\perp$. Therefore, it is enough to consider $\theta \in (0, \frac{\pi}{2})$.

The following properties exhibit the geometry of the above-introduced subsets.

(1) For any $\theta \in (0,\frac{\pi}{2})$, the set $\mathbf{C}_\theta$ is the intersection of $\mathbb{S}^3$ with a cone in $\mathbb{R}^4 = \mathbb{C}^2$. More precisely, in a suitable coordinate system (z_1, z_2) in $\mathbb{C}^2$, we can check that $\mathbf{C}_\theta = \mathbb{S}^3 \bigcap \Omega_\theta$, where

$$\Omega_\theta = \{\zeta = (z_1,z_2) \in \mathbb{C}^2 : |z_1|^2\sin^2\theta - |z_2|^2\cos^2\theta = 0\}.$$

(2) Furthermore, $\mathbf{C}_\theta$ can be identified with the following torus

$$\mathbf{C}_\theta = \{\zeta = (z_1,z_2) \in \mathbb{C}^2 : |z_1| = \cos\theta, |z_2| = \sin\theta\}.$$

(3) For any $\theta \in (0,\frac{\pi}{2})$, any great circle $\mathbf{C}$ and $\zeta \in \mathbf{C}_\theta$, there exist exactly two great circles, $\mathbf{C}'$ and $\mathbf{C}''$, that are Clifford parallel to $\mathbf{C}$ and contain ζ. This shows that the Clifford parallelism is not an equivalence relation.

Though the Clifford parallelism is not an equivalence relation, it can be decomposed into two equivalence relations. Let us sketch how to do it. Given a great circle, $\mathbf{C}$, we have $\mathbf{C}^{\perp}$, so that the planes $\mathbf{P}, \mathbf{P}^{\perp}$, through the origin containing these two great circles satisfy $\mathbf{P} \oplus \mathbf{P}^{\perp} = \mathbb{R}^4$. We also fix an orientation on $\mathbf{P}$ and on $\mathbf{P}^{\perp}$ to get the canonical orientation in $\mathbb{R}^4$ according to the above decomposition. Next, define subgroups of $\mathbf{O}^+(\mathbf{P}) \times \mathbf{O}^+(\mathbf{P}^{\perp}) \subset \mathbf{O}^+(\mathbb{R}^4)$ by

$$\mathbf{G}_{\mathbf{C}}^{+} = \{(\omega; f_+ \circ \omega \circ f_+^{-1}) : \omega \in \mathbf{O}^+(\mathbf{P})\},$$
$$\mathbf{G}_{\mathbf{C}}^{-} = \{(\omega; f_- \circ \omega \circ f_-^{-1}) : \omega \in \mathbf{O}^+(\mathbf{P})\}$$

where $f_+ \in \mathbf{Iso}^+(\mathbf{P}, \mathbf{P}^{\perp})$ (resp. $f_- \in \mathbf{Iso}^-(\mathbf{P}, \mathbf{P}^{\perp})$) is an orientation preserving (resp. non-preserving) isometry. It should be noticed that this construction does not depend on f_+ (or f_-) since $\mathbf{O}^+(\mathbf{P})$ is abelian. Now, an orbit under the $\mathbf{G}_C^+$-action is a great circle, say $\mathbf{C}'$, that is called a Clifford parallel to $\mathbf{C}$ of the first-kind while the second-kinds of Clifford parallels, $\mathbf{C}''$, are obtained via the second subgroup. These are two equivalence relations which are denoted by $\mathbf{C} \|^+ \mathbf{C}'$ and $\mathbf{C} \|^- \mathbf{C}''$. Furthermore, we have the following facts.

(1) The condition $\mathbf{C} \| \tilde{\mathbf{C}}$ is equivalent to either $\mathbf{C} \|^+ \tilde{\mathbf{C}}$ or $\mathbf{C} \|^- \tilde{\mathbf{C}}$.
(2) For each $\zeta \in \mathbb{S}^3$, there exist two great circles through ζ that are Clifford parallel to $\mathbf{C}$, one of the first-kind, $\mathbf{C}'$, and one of the second, $\mathbf{C}''$. Furthermore, $\mathbf{C}' \neq \mathbf{C}''$ if $\zeta \in \mathbb{S}^3 \setminus (\mathbf{C} \bigcup \mathbf{C}^{\perp})$.

Clifford parallel great circles are nicely related with Villarceau circles through a suitable stereographic projection. We take $\zeta_o \in \mathbb{S}^3 \subset \mathbb{R}^4$ and consider the stereographic projection $E_o : \mathbb{S}^3 \setminus \{\zeta_o\} \to \mathbb{R}^3$ which, as is well known, is the restriction of an inversion in $\mathbb{R}^4$ with pole ζ_o. Now, fix a great circle, say $\mathbf{C}$, going through ζ_o. In $\mathbb{R}^3$, we choose a coordinate system, $\{x, y, z\}$, such that the z-axis is $E_o(\mathbf{C} \setminus \{\zeta_o\})$ and then, $E_o(\mathbf{C}^{\perp})$ is the unit circle in the $\{x, y\}$-plane.

In that setting, it is not difficult to see that $\mathbf{T}_\theta = E_o(\mathbf{C}_\theta)$, $\theta \in (0, \frac{\pi}{2})$, is a revolution torus around $E_o(\mathbf{C} \setminus \{\zeta_o\})$ in $\mathbb{R}^3$. Furthermore, up to similarities, every revolution torus in $\mathbb{R}^3$ is of the form $E_o(\mathbf{C}_\theta)$ for a suitable value $\theta \in (0, \frac{\pi}{2})$. Now, both families of Villarceau circles in $\mathbf{T}_\theta = E_o(\mathbf{C}_\theta)$ are obtained as images under the stereographic projection, E_o, of the two kinds of great circles in $\mathbf{C}_\theta$ that are Clifford parallel to $\mathbf{C}$. And so, all Villarceau circles in $\mathbb{R}^3 \setminus$ (z-axis) can be described as follows.

$$\mathcal{F}_1 = \{\Upsilon(t)\} = \{E_o(\mathbf{C}') : \mathbf{C}' \text{ is first-kind Clifford parallel to } \mathbf{C}\},$$

and

$$\mathcal{F}_2 = \{\Xi(t)\} = \{E_o(\mathbf{C}'') : \mathbf{C}'' \text{ is second-kind Clifford parallel to } \mathbf{C}\}.$$

From now on, we will refer to these circles as first- or second-kind Villarceau circles according to whether they lie in $\mathcal{F}_1$ or $\mathcal{F}_2$, respectively.

Notice that in $\mathcal{F}_1$ and in $\mathcal{F}_2$ we have included a circle that is not a Villarceau circle in a revolution torus around the z-axis, this circle is $E_o(\mathbf{C}^{\perp})$. From now on we will treat this circle as a Villarceau circle.

4. Villarceau circles as orbits

In the appropriate coordinate system, the action of $\mathbf{G}_{\mathbf{C}}^{+}$ on $\mathbb{S}^3$ is the usual one described as

$$\mathbf{G}_{\mathbf{C}}^{+} \times \mathbb{S}^3 \to \mathbb{S}^3, \qquad (\varphi_t, \zeta) \mapsto \varphi_t(\zeta) = \mathrm{e}^{\mathrm{i}t} \cdot \zeta.$$

Hence, the orbits under this action, that is to say, the first-kind Clifford parallel great circles to $\mathbf{C}$, are nothing but the fibres of the usual Hopf map, $\Pi : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$, $\Pi(z_1, z_2) = (z_1 \bar{z_2}, \frac{1}{2}(|z_1|^2 - |z_2|^2))$, where $\bar{z_2}$ is the complex conjugate of z_2. To simplify, we put $\mathbf{G}_{\mathbf{C}}^{+} = \{\varphi_t : t \in \mathbb{R}\}$.

This can be projected down to $\mathbb{R}^3$. Indeed, since $\mathbf{C}$ is the orbit through $\zeta_o \in \mathbf{C}$ and we have chosen $E_o(\mathbf{C})$ to be the z-axis in $\mathbb{R}^3$, we have a group of preserving orientation conformal maps in $\mathbb{R}^3 \setminus$ (z-axis) associated with $\mathbf{C}$, and defined by

$$\mathbf{H}_{\mathbf{C}}^{+} = E_o \circ \mathbf{G}_{\mathbf{C}}^{+} \circ E_o^{-1} = \{\psi_t = E_o \circ \varphi_t \circ E_o^{-1} : t \in \mathbb{R}\}.$$

In this setting, the orbits in $\mathbb{R}^3 \setminus$ (z-axis) associated with $\mathbf{H}_{\mathbf{C}}^{+}$ are just the first-kind Villarceau circles over a family of revolution tori around the z-axis.

Notice that, given a pair of first-kind Villarceau circles, say γ_1 and γ_2, then $\gamma_1 = E_o(\mathbf{C}_1)$ and $\gamma_2 = E_o(\mathbf{C}_2)$ for certain great circles that satisfy $\mathbf{C}\|^+ \mathbf{C}_1$, $\mathbf{C}\|^+ \mathbf{C}_2$ and so $\mathbf{C}_1 \|^+ \mathbf{C}_2$. In other words, those Villarceau circles are images, via a stereographic projection, of two Hopf fibres. However, they can lie on either the same revolution torus or two different revolution tori. The former occurs if $\mathrm{d}(\mathbf{C}, \mathbf{C}_1) = \mathrm{d}(\mathbf{C}, \mathbf{C}_2)$ while the latter happens if $\mathrm{d}(\mathbf{C}, \mathbf{C}_1) \neq \mathrm{d}(\mathbf{C}, \mathbf{C}_2)$.

Next, we deal with second-kind Clifford parallel circles. As before, in a suitable coordinate system, we put $\mathbf{G}^-_{\mathbf{C}} = \{\chi_t : t \in \mathbb{R}\}$ and the action of $\mathbf{G}^-_{\mathbf{C}}$ on $\mathbb{S}^3$ is described as

$$\mathbf{G}^-_{\mathbf{C}} \times \mathbb{S}^3 \to \mathbb{S}^3, \qquad (\chi_t, (z_1, z_2)) \mapsto \chi_t(z_1, z_2) = (\mathrm{e}^{\mathrm{i}t} z_1, \mathrm{e}^{-\mathrm{i}t} z_2).$$

Similar to the usual Hopf map, the projection map to the quotient space is

$$\Pi_- : \mathbb{S}^3 \to \mathbb{S}^2\left(\frac{1}{2}\right), \qquad \Pi_-(z_1, z_2) = \left(z_1 z_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)\right).$$

As before, the fibres of Π_- are nothing but the second-kind Clifford parallel circles to $\mathbf{C}$. Again, let

$$\mathbf{H}^-_{\mathbf{C}} = E_o \circ \mathbf{G}^-_{\mathbf{C}} \circ E_o^{-1} = \{E_o \circ \chi_t \circ E_o^{-1} : t \in \mathbb{R}\}$$

be the group of conformal maps that leave invariant the second-kind Villarceau circles over a family of revolution tori around the z-axis.

Next, we just need to see that the isometry $J : \mathbb{S}^3 \to \mathbb{S}^3$, $J(z_1, z_2) = (z_1, \bar{z_2})$ lets us construct the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{J} & \mathbb{S}^3 \\ \Pi \downarrow & & \downarrow \Pi_- \\ \mathbb{S}^2\left(\frac{1}{2}\right) & \xrightarrow{Id} & \mathbb{S}^2\left(\frac{1}{2}\right). \end{array}$$

Therefore, up to small changes, we can reduce all computations to the case of first-kind Villarceau circles.

5. Solitons in the $\mathbf{NSM}_2$ admitting a Villarceau foliation

We denote by $\mathbf{H}_{\mathbf{C}}$ either $\mathbf{H}^+_{\mathbf{C}}$ or $\mathbf{H}^-_{\mathbf{C}}$. In this section, we deal with the following

Problem 4. To obtain all the solitons of the $\mathbf{NSM}_2$ with boundary that are invariant under $\mathbf{H}_{\mathbf{C}}$.

In other words, to obtain all the solutions of the field equations that are foliated by Villarceau circles. From now on, we consider symmetric solutions. We solve completely this problem with a geometric argument that involves the following steps

- Admissible boundary conditions.
- Reduction of symmetry.
- Using the conformal invariance and the Hopf map to reduce variables.
- The solutions come from clamped elasticae in the two-sphere.

For the sake of simplicity, we just make the computations for the case of first-kind Villarceau circles. Once we have computed the clamped elasticae in the two-sphere, since we have two maps $\Pi \circ E_0^{-1}$, $\Pi_- \circ E_0^{-1} : \mathbb{R}^3 \setminus (z\text{-axis}) \to \mathbb{S}^2(\frac{1}{2})$, we obtain the two families of solutions.

5.1. Admissible boundary conditions

It is clear that the admissible boundary conditions, for obtaining $\mathbf{H}^+_{\mathbf{C}}$-invariant solutions, must be, themselves, $\mathbf{H}^+_{\mathbf{C}}$-invariant. This yields to the following immediate first constraints

(1) $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is made up of first-kind Villarceau circles.

(2) N_o is a unit vector field along Γ with $\langle \Gamma', N_o\rangle = 0$, which is constructed as follows. Fix points, $p_i \in \gamma_i$, $1 \le i \le n$, at each component of Γ. So, $\gamma_i(t) = \psi_t(p_i)$. Next, we choose unit vectors, $n_i \in T_{p_i}\mathbb{R}^3$, with $\langle n_i, \gamma_i'(0)\rangle = 0$. Finally, we define

$$N_o(\gamma_i(t)) = N_o(\psi_t(p_i)) = \frac{(\mathrm{d}\psi_t)_{p_i}(n_i)}{\|(\mathrm{d}\psi_t)_{p_i}(n_i)\|}.$$

Obviously, the constructed vector field, N_o, along Γ satisfies $\mathrm{d}\psi_t(N_o) = \lambda N_o$, for a suitable $\lambda > 0$, and so it does not depend on the starting time $t \in \mathbb{R}$. Notice also that we have a third vector field, say ν, along Γ determined by $\Gamma'(p) \wedge \nu(p) = N_o(p)$, $\forall p \in \Gamma$, which is also invariant under $\mathbf{H}^+_{\mathbf{C}}$.

However, those conditions are not sufficient to guarantee the existence of symmetric solutions. A deeper analysis yields a more subtle constraint. In fact, since we are looking for solutions that are foliated by Villarceau circles (and of course they are connected), then their images in the corresponding space of orbits must be connected curves. In other words, the transversal submanifold to the foliation is a connected curve. This automatically implies that Γ is constituted by exactly a pair of orbits. Therefore, the admissible boundary conditions are the above-mentioned ones with $n = 2$.

In this setting, given the boundary data ($\Gamma = \{\gamma_1, \gamma_2\}, N_o$), we choose $M = \mathbb{S}^1 \times [a_1, a_2]$ a surface with boundary $\partial M = (\mathbb{S}^1 \times \{a_1\}) \bigcup (\mathbb{S}^1 \times \{a_2\})$ and consider $I_\Gamma(M, \mathbb{R}^3)$, the space of immersions $\phi : M \to \mathbb{R}^3$, that satisfy the following boundary conditions

(1) $\phi(\mathbb{S}^1 \times \{a_1\}) = \gamma_1, \phi(\mathbb{S}^1 \times \{a_2\}) = \gamma_2$,
(2) $\mathrm{d}\phi_q(T_q M) \perp N_o(\phi(q))$, for all $q \in \partial M$.

Then, we have the $\mathbf{NSM}_2$ with boundary $(I_\Gamma(M, \mathbb{R}^3), \mathcal{S})$ and the problem is to obtain all its solitons that preserve the symmetry of the boundary, in other words, to get all the solutions of the associated field equations that are $\mathbf{H}^+_{\mathbf{C}}$-invariant. It should be noticed that this invariance geometrically means that the solutions we are looking for are foliated by first-kind Villarceau circles.

5.2. Reduction of symmetry

In the above-described setting, the group $\mathbf{H}^+_{\mathbf{C}}$ naturally acts on $I_\Gamma(M, \mathbb{R}^3)$. In fact, we have

$$\mathbf{H}^+_{\mathbf{C}} \times I_\Gamma(M, \mathbb{R}^3) \to I_\Gamma(M, \mathbb{R}^3), \qquad (f, \phi) \mapsto f \circ \phi.$$

Furthermore, the $\mathbf{NSM}_2$ Lagrangian $\mathcal{S} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}$ is obviously $\mathbf{H}^+_{\mathbf{C}}$-invariant, i.e., $\mathcal{S}(f \circ \phi) = \mathcal{S}(\phi)$, $\forall f \in \mathbf{H}^+_{\mathbf{C}}$ and $\forall \phi \in I_\Gamma(M, \mathbb{R}^3)$.

The principle of symmetric criticality [23] can be applied in this framework, working in the following way. Let Σ be the space of immersions, $\phi \in I_\Gamma(M, \mathbb{R}^3)$, which are $\mathbf{H}^+_{\mathbf{C}}$-invariant, that is $f \circ \phi = \phi$, $\forall f \in \mathbf{H}^+_{\mathbf{C}}$. We will refer to these immersions as symmetric points and notice that we are looking for the solitons in Σ. Then, a symmetric point, $\phi \in \Sigma$, is a solution in the $\mathbf{NSM}_2$ with boundary $(I_\Gamma(M, \mathbb{R}^3), \mathcal{S})$ if and only if it is a critical point of $\mathcal{S} : \Sigma \to \mathbb{R}$. In other words, the $\mathbf{H}^+_{\mathbf{C}}$-invariant solutions of the field equations coincide with the solutions of the $\mathbf{H}^+_{\mathbf{C}}$-reduced field equations.

To compute this restriction, first, we need to identify the space Σ. Therefore, we consider the following map

$$\Phi = \Pi \circ E_o^{-1} : \mathbb{R}^3 \setminus (z\text{-axis}) \to \mathbb{S}^2\left(\frac{1}{2}\right) \setminus \{m\},$$

where $m \in \mathbb{S}^2(\frac{1}{2})$ stands for the image of the great circle $\mathbf{C}$ under the Hopf map, $\Pi : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$. We take the two-sphere of radius $\frac{1}{2}$ in order for Π to be a Riemannian submersion, obtaining that Φ is a conformal submersion.

Now, admissible boundary conditions behave, via that map, as follows. First of all, we observe that $\Gamma = \{\gamma_1, \gamma_2\}$ is applied, via E_o^{-1}, in a pair of Hopf fibres, $(\mathbf{C}_1, \mathbf{C}_2)$. Hence $\Phi(\Gamma) = \{m_1, m_2\}$ is a couple of points in $\mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$. Furthermore,

$$\left\{\vec{u}_i = \frac{\mathrm{d}\Phi_{p_i}(\nu(p_i))}{\|\mathrm{d}\Phi_{p_i}(\nu(p_i))\|}; \vec{w}_i = \frac{\mathrm{d}\Phi_{p_i}(N_o(p_i))}{\|\mathrm{d}\Phi_{p_i}(N_o(p_i))\|}\right\}, \quad 1 \le i \le 2,$$

constitutes an orthonormal frame in $T_{m_i}\mathbb{S}^2(\frac{1}{2})$, which does not depend on the chosen points $p_i \in \gamma_i$, $1 \le i \le 2$.

For any regular curve, $\alpha : [s_1, s_2] \to \mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$, with $\alpha(s_i) = m_i$ and $\alpha'(s_i) = \vec{u}_i$, $1 \le i \le 2$, we denote by $M_\alpha = \Phi^{-1}(\alpha)$ the surface obtained as the stereographic projection of the Hopf tube, $\Pi^{-1}(\alpha)$, on α. In particular, M_α is foliated by first-kind Villarceau circles and so it is invariant under the $\mathbf{H}^+_{\mathbf{C}}$-action. Therefore, the immersion $\phi \in I_\Gamma(M, \mathbb{R}^3)$ such that $\phi(M) = M_\alpha$ obviously lies in Σ. Conversely, if $\phi \in \Sigma$, then we can regard its image, $\phi(M)$, as a surface which is foliated by first-kind Villarceau circles and so it is the image, under the stereographic projection, of a Hopf tube on a certain curve, with the obvious first-order boundary data. Hence, the space Σ can be identified with the following class of surfaces

$$\Sigma \equiv \left\{ M_\alpha = \Phi^{-1}(\alpha)/\alpha : [s_1, s_2] \to \mathbb{S}^2\left(\frac{1}{2}\right) \setminus \{m\}, \alpha(s_i) = m_i, \alpha'(s_i) = \vec{u}_i, 1 \le i \le 2 \right\}.$$

We have proved that the symmetric solutions of $(I_\Gamma(M, \mathbb{R}^3), (\Gamma, N_o), \mathcal{S})$, are just the critical points of $\mathcal{S}/_\Sigma : \Sigma \to \mathbb{R}$. Then, we need to characterize the critical points of this action.

5.3. Using the conformal invariance and the Hopf map to reduce variables

As seen in Section 2, the $\mathbf{NSM}_2$ is equivalent to the following Willmore problem with boundary [29]

$$\mathcal{W}(\phi) = \int_M H^2_\phi \mathrm{d}A_\phi + \int_{\phi(\partial M)} k^\phi \mathrm{d}s.$$

In particular, the class of soliton solutions in the $\mathbf{NSM}_2$ coincides with the class of Willmore soliton surfaces with boundary. Furthermore, since the Willmore functional is invariant under the action of $\mathbf{H}^+_{\mathbf{C}}$ on $I(M, \mathbb{R}^3)$, then both problems are also equivalent when reduced via $\mathbf{H}^+_{\mathbf{C}}$. In other words, to compute all the symmetric solutions for $\mathbf{NSM}_2$, we only need to compute the critical points of $\mathcal{W} : \Sigma \to \mathbb{R}$.

The next idea is to exploit the extrinsic conformal invariance of the Willmore model with boundary. In particular, $E_o : \mathbb{S}^3 \setminus \{\zeta_o\} \to \mathbb{R}^3$ is a conformal transformation, which implies that

$$\mathcal{W}(M_\alpha) = \mathcal{W}(\Phi^{-1}(\alpha)) = \bar{\mathcal{W}}(\Pi^{-1}(\alpha)) = \int_M (\bar{H}^2_\alpha + \bar{R}_\alpha) \mathrm{d}\bar{A}_\alpha + \int_{\partial M} \bar{k}^\alpha \mathrm{d}s,$$

where $\bar{\mathcal{W}}$ stands for the Willmore functional associated with the unit round metric in the three-sphere and so, $\bar{R}_\alpha$ is the sectional curvature along $\phi \in I(M, \mathbb{S}^3)$ with $\phi(M) = \Pi^{-1}(\alpha)$.

In this case, $R_\alpha = 1$ because it is a part of the sectional curvature in a unit round sphere. On the other hand, the boundary of any of those symmetric immersions giving the Hopf tubes, $\Pi^{-1}(\alpha)$, is $\partial \Pi^{-1}(\alpha) = \{\mathbf{C}_1, \mathbf{C}_2\}$. That is, the boundary is made up of two great circles in $\mathbb{S}^3$, which are geodesics and so $\bar{k}^\alpha = 0$. Therefore, we have

$$\mathcal{W}(M_\alpha) = \bar{\mathcal{W}}(\Pi^{-1}(\alpha)) = \int_M (\bar{H}^2_\alpha + 1) \mathrm{d}\bar{A}_\alpha.$$

Now, the mean curvature of complete lifts of curves in a Riemannian submersion with totally geodesics fibers was computed in [6]. In particular, the mean curvature of a Hopf tube, $\Pi^{-1}(\alpha)$, is $\bar{H}_\alpha = \frac{1}{2}\kappa$, where κ is the curvature function of the curve α in the two-sphere. Consequently, we have

$$\mathcal{W}(M_\alpha) = \bar{\mathcal{W}}(\Pi^{-1}(\alpha)) = \frac{\pi}{2} \int_\alpha (\kappa^2 + 4) \, \mathrm{d}s. \tag{1}$$

5.4. The solutions come from clamped elasticae in the two-sphere

Hence, the problem of searching the symmetric solutions of the $\mathbf{NSM}_2$ with boundary, $(I_\Gamma(M, \mathbb{R}^3), (\Gamma, N_o), \mathcal{S})$ is reduced to that for clamped elastica in the once-punctured two-sphere. To be precise, in $\mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$ we choose two points, m_1 and m_2, unit vectors $\vec{u}_i \in T_{m_i}\mathbb{S}^2$ and the space of clamped curves, $\Lambda = \{\alpha : [s_1, s_2] \to \mathbb{S}^2(\frac{1}{2}) \setminus \{m\}/\alpha(s_i) = m_i, \alpha'(s_i) = \vec{u}_i, 1 \le i \le 2\}$, and then the variational problem associated with the total elastic energy, $\mathcal{E} : \Lambda \to \mathbb{R}$, given by

$$\mathcal{E}(\alpha) = \int_\alpha (\kappa^2 + 4) \, \mathrm{d}s.$$

Theorem 5. *Let $\Gamma = \{\gamma_1, \gamma_2\}$ be a pair of Villarceau circles in $\mathcal{F}_1$ or $\mathcal{F}_2$ and N_o a unit normal vector field along Γ. Then $\phi \in I_\Gamma(M, \mathbb{R}^3)$ is a soliton of* $\mathbf{NSM}_2$*, with boundary conditions (Γ, N_o), which is foliated by Villarceau circles if, and only if, either:*

(1) *$\phi(M)$ is the stereographic projection of a Hopf tube on a clamped elastic curve in the once-punctured two-sphere.*
(2) *$\phi(M)$ is the stereographic projection of the complete lifting, via the mapping $\Pi_- : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$, of a clamped elastic curve in the once-punctured two-sphere.*

To conclude the algorithm, recall the description of the moduli space of clamped elastic curves in a two-sphere with radius $\frac{1}{2}$ (see [11] and [17] for more details and terminology). Thus, we obtain the first variation formula for $\mathcal{E}$

$$\delta\mathcal{E}(\alpha)[W] = \int_\alpha \langle \Omega(\alpha), W\rangle \mathrm{d}s + [\mathcal{R}(\alpha, W)]_{s_1}^{s_2},$$

where $\Omega(\alpha)$ and $\mathcal{R}(\alpha, W)$ stand for the Euler–Lagrange and boundary operators, respectively, given by

$$\Omega(\alpha) = 2\nabla_T^3 T + \nabla_T[(3\kappa^2 - 4)T] + 8\nabla_T T,$$

and

$$\mathcal{R}(\alpha, W) = 2\langle \nabla_T W, \nabla_T T\rangle - \langle W, 2\nabla_T^2 T + (3\kappa^2 - 4)T\rangle,$$

where ∇ denotes the Levi-Civita connection of $\mathbb{S}^2(\frac{1}{2})$, T is the unit tangent vector field of α and $W \in T_\alpha \Lambda$.

On the other hand, we can make the following computations along a curve, $\bar{\alpha}$, in Λ with first-order data (α, W)

$$W = \mathrm{d}\bar{\alpha}(\partial_r), \qquad \nabla_T W = \vartheta T + \mathrm{d}\bar{\alpha}(\partial_r T),$$

where $\vartheta = \partial_r(\log \|\frac{\partial\bar{\alpha}}{\partial t}(t, r)\|)$. Then, we evaluate these formulas along the curve α by making $r = 0$ and use the first-order boundary data to obtain the following values at the endpoints

$$W(s_i) = 0, \qquad \nabla_T W(s_i) = \vartheta(s_i)\vec{u}_i, \quad 1 \le i \le 2.$$

As a consequence, the boundary operator drops out, $[\mathcal{R}(\alpha, W)]_{s_1}^{s_2} = 0$.

Then, $\alpha \in \Lambda$ is a critical point of the variational problem $\mathcal{E} : \Lambda \to \mathbb{R}$ if, and only if, $\Omega(\alpha) = 0$ and it happens if, and only if, the curvature function of α is a solution of the following second-order differential equation

$$2\frac{\mathrm{d}^2\kappa}{\mathrm{d}s^2} + \kappa(\kappa^2 + 4) = 0.$$

These curves will be called clamped elasticae in the two-sphere, $\mathbb{S}^2(\frac{1}{2})$ and we will briefly describe them using the free boundary case. First, notice that this equation admits the obvious constant solution $\kappa = 0$ (geodesics). Those clamped geodesics provide, when lifting via the Hopf map, Clifford tubes with boundary in $\mathbb{S}^3$. The stereographic projection of those Clifford tubes give tubes with boundary in anchor rings with ratio $\sqrt{2}$ as solitons of the $\mathbf{NSM_2}$ with boundary, $(I_\Gamma(M, \mathbb{R}^3), (\Gamma, N_o), \mathcal{S})$ (these solutions were already given in [7]).

When searching for non-constant solutions, we realize that the equation is readily solved by observing the following identity

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\,\mathrm{cn}(u, \rho) = -2\rho^2\mathrm{cn}^3(u, \rho) + (2\rho^2 - 1)\,\mathrm{cn}(u, \rho),$$

where $\mathrm{cn}(u, \rho)$ is the elliptic cosine of Jacobi [11].

Replacing u by $\lambda(s - s_0)$ and making an identification of constants, we readily obtain the general solution of the equation in the form:

$$\kappa(s) = \mathcal{C}\mathrm{cn}(\lambda(s - s_0), \rho),$$

where λ and s_0 are arbitrary constants, and where ρ and $\mathcal{C}$ are determined as follows:

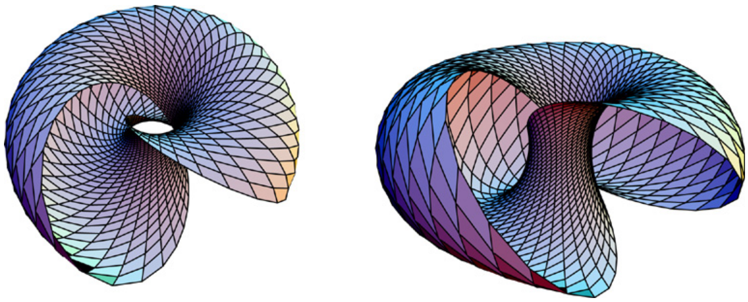
$$\rho^2 = \frac{\lambda^2 - 2}{2\lambda^2}, \qquad \mathcal{C}^2 = 2(\lambda^2 - 2).$$

Since ρ and $\mathcal{C}$ are thus functions of λ, it is clear that either of them, but not both, may be chosen arbitrarily. Taking $\lambda^2 \geq 2$, then ρ^2 and $\mathcal{C}^2$ are positive. Also, $\mathrm{cn}(-u,\rho) = \mathrm{cn}(u,\rho)$, so we can take $\lambda \geq \sqrt{2}$.

Finally the curvature is $\kappa(s) = \sqrt{2(\lambda^2-2)}\mathrm{cn}(\lambda(s-s_0), \frac{\sqrt{\lambda^2-2}}{\sqrt{2}\lambda})$, where $\lambda \geq \sqrt{2}$ is a constant.

This family of elasticae belong to a wider class known as wavelike. This name comes from the fact that each of these elasticae oscillates along an axial geodesic in the two-sphere.

The following two pictures illustrate the shape of our solitons. In fact, they show the same surface from two different points of view. The foliation by Villarceau circles can be seen clearly.



6. The topological charge

The above-obtained solitons of $\mathbf{NSM}_2$ carry topological charges, which can be holographically determined, via the Gauss–Bonnet formula, from the boundary conditions. Hence if $\phi \in I_\Gamma(M, \mathbb{R}^3)$ is a soliton of $\mathbf{NSM}_2$ with boundary conditions $(\Gamma = \{\gamma_1, \gamma_2\}, N_0)$, then its topological charge is

$$Q(\phi) = \int_M G_\phi \mathrm{d}A_\phi = -\mathcal{L}(\phi) = -\sum_{i=1}^{2}\int_{\gamma_i} k_i \,\mathrm{d}s.$$

Therefore, we need to compute the right hand side in the above expression where γ_1 and γ_2 are Villarceau circles. We will restrict ourselves to the case where γ_1 and γ_2 are first-kind Villarceau circles, since the computations for second-kind Villarceau circles are similar.

It is clear that any Villarceau circle in $\mathbb{R}^3$ intersects in exactly one point the half-plane $K = \{(x,0,z) \in \mathbb{R}^3 : x > 0\}$. Also, we know that two Villarceau circles of the same kind do not intersect. Then, any point of K determines two Villarceau circles, one of each kind. If

$$E_o^{-1}(x,0,z) = (x_1, 0, x_2, y_2) = \left(\frac{2x}{x^2+z^2+1}, 0, \frac{2z}{x^2+z^2+1}, \frac{x^2+z^2-1}{x^2+z^2+1}\right),$$

then, the first-kind Villarceau circle, $\gamma : [-\pi,\pi] \to \mathbb{R}^3$, passing through the point $p = (x,0,z)$, namely with $\gamma(0) = p$, is given by

$$\begin{aligned}\gamma(t) &= E_o(\mathrm{e}^{\mathrm{i}t}x_1, \mathrm{e}^{\mathrm{i}t}(x_2 + \mathrm{i}y_2)) \\ &= \frac{1}{1 - x_2\sin t - y_2\cos t}(x_1\cos t, x_1\sin t, x_2\cos t - y_2\sin t).\end{aligned}$$

Consequently, in the above parametrization, the speed of the Villarceau circle is

$$\|\gamma'(t)\| = \frac{1}{1 - x_2\sin t - y_2\cos t}, \tag{2}$$

and so the length and the radius of the Villarceau circle passing through the point $p = (x,0,z)$ are, respectively

$$L = \int_{-\pi}^{\pi}\|\gamma'(t)\|\mathrm{d}t = \pi\frac{1+\|p\|^2}{x}, \qquad r = \frac{1+\|p\|^2}{2x}.$$

Now, we suppose that γ is one of the Villarceau circles in $\Gamma = \{\gamma_1, \gamma_2\}$. It should be noticed that, though γ has obviously constant curvature in $\mathbb{R}^3$, its curvature function in $\phi(M)$ is not constant in general. If the Villarceau circle is arc length parametrized its curvature function in the soliton is given by

$$k(s) = -2\left(\frac{1}{1+\|\gamma(s)\|^2}\right)\langle\gamma(s), \nu(s)\rangle, \tag{3}$$

where $\nu(s)$ is determined by $N_0(s) = \gamma'(s) \wedge \nu(s)$. This formula is a consequence of the following

Lemma 6. *Let* $\mathbf{N} = E_o(\mathcal{T})$ *be the stereographic projection of a Hopf tube and* $\gamma = E_o(\tilde{\mathbf{C}})$ *a Villarceau circle in* $\mathbf{N}$. *The curvature function of* γ *in* $\mathbf{N}$ *is given by* (3), *where* $\nu(s)$ *plays the role of unit normal vector of* γ *in* $\mathbf{N}$.

Proof. Denote by g_1 the flat metric on $\mathcal{T}$ which is induced by that usual in the unit three-sphere. Now the metric $g_2 = E_o^*(\langle\cdot,\cdot\rangle)$ is conformal to the above metric. Namely $g_2 = f\, g_1$ where f is the restriction to $\mathcal{T}$ of the well known conformal factor associated with the stereographic projection, E_o. The curvature function of γ in $\mathbf{N}$ is just that of $\tilde{\mathbf{C}}$ in $(\mathcal{T}, g_2)$ and it can be nicely obtained, as is well known, in terms of both the curvature function of $\tilde{\mathbf{C}}$ in $(\mathcal{T}, g_1)$ and the normal variation of the conformal factor. Since $\tilde{\mathbf{C}}$ is a geodesic in $(\mathcal{T}, g_1)$, we obtain

$$k(s) = -\frac{1}{2(f\circ E_o^{-1})(\gamma(s))}\langle\nabla(f\circ E_o^{-1})(\gamma(s)), \nu(s)\rangle,$$

where ∇ stands for the $\langle\cdot,\cdot\rangle$-gradient. However,

$$(f\circ E_o^{-1})(p) = \left(\frac{1+\|p\|^2}{2}\right)^2,$$

and so

$$\nabla(f\circ E_o^{-1})(\gamma(s)) = \left(1+\|\gamma(s)\|^2\right)\gamma(s),$$

which proves the lemma. □

Remark 7. We should point out that all the computations in this lemma are also valid when $\mathcal{T}$ is a lift of a curve in $\mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$ by the map Π_-, and hence for second-kind Villarceau circles.

Now, the total curvature of a first-kind Villarceau circle in a soliton is computed using (2) and (3) as follows

$$\int_\gamma k(s)\,\mathrm{d}s = \int_{-\pi}^{\pi} k(t)\|\gamma'(t)\|\mathrm{d}t = -\int_{-\pi}^{\pi}\langle\gamma(t), \nu(t)\rangle\mathrm{d}t. \tag{4}$$

According to the admissible boundary conditions (see Section 5.1), in the soliton, the unit normal vector field along $\gamma(t)$ can be obtained from $\nu(0) = (\nu_1, \nu_2, \nu_3)$, by

$$\nu(t) = \frac{\mathrm{d}\psi_t(\nu(0))}{\|\mathrm{d}\psi_t(\nu(0))\|} = \frac{\mathrm{d}E_o(\mathrm{e}^{\mathrm{i}t}\mathrm{d}E_o^{-1}(\nu(0)))}{\|\mathrm{d}E_o(\mathrm{e}^{\mathrm{i}t}\mathrm{d}E_o^{-1}(\nu(0)))\|}.$$

The integrand in (4) can be computed, by means of a direct long computation that involves the conformal nature of E_o, in terms of the following data

$$E_o^{-1}(\gamma(0)) = (x_1, 0, x_2, y_2), \qquad \text{and} \qquad \mathrm{d}E_o^{-1}(\nu(0)) = (u_1, v_1, u_2, v_2),$$

with $x_1u_1 + x_2u_2 + y_2v_2 = x_1v_1 - y_2u_2 + x_2v_2 = 0$ because $(u_1, v_1, u_2, v_2) \in T_{(x_1,0,x_2,y_2)}\mathbb{S}^3$ and it must be a horizontal vector. The result is

$$\langle\gamma(t), \nu(t)\rangle = \frac{u_2\,\sin t + v_2\,\cos t}{(1-y_2)(1-y_2\,\cos t - x_2\,\sin t)}. \tag{5}$$

It should be noticed that for any N_0, γ_i is not a geodesic in the soliton $\phi \in I_\Gamma(M, \mathbb{R}^3)$. In particular, there are no Villarceau circles that are geodesics in the solitons. However, the total curvature of a Villarceau circle in a soliton can vanish. In fact, choose the Villarceau circle through the point $(1, 0, 0) \in \mathbb{R}^3$, for any N_0 its total curvature is

$$\int_\gamma k(s)\,\mathrm{d}s = -\int_{-\pi}^{\pi} \langle \gamma(t), \nu(t)\rangle \mathrm{d}t = -\int_{-\pi}^{\pi} (u_2 \sin t + v_2 \cos t)\,\mathrm{d}t = 0.$$

The total curvature of any Villarceau circle in a soliton can be computed integrating (5). As we know what happens when $x_1 = 1$, let us now assume that $x_1 < 1$, so $x_2^2 + y_2^2 > 0$, and let us make the change $\tau = \tan \frac{t}{2}$ to obtain

$$\int_\gamma \langle \gamma(t), \nu(t)\rangle \mathrm{d}s = \frac{2}{1-y_2} \int_{-\infty}^{\infty} \frac{-v_2\tau^2 + 2u_2\tau + v_2}{\left((1+y_2)\tau^2 - 2x_2\tau + 1 - y_2\right)(1+\tau^2)} \mathrm{d}\tau.$$

This integral can be evaluated using the partial fractions method. Notice that the first factor in the denominator is actually quadratic because $x_2^2 + y_2^2 - 1 < 0$. Then, we have

$$\int_\gamma \langle \gamma(t), \nu(t)\rangle \mathrm{d}s = \frac{2}{1-y_2} \int_{-\infty}^{\infty} \left[\frac{A\tau + B}{1+\tau^2} + \frac{C\tau + D}{(1+y_2)\tau^2 - 2x_2\tau + 1 - y_2}\right] \mathrm{d}\tau.$$

Then, we get

$$\begin{aligned}\int_\gamma \langle \gamma(t), \nu(t)\rangle \mathrm{d}s =\ & \frac{2\pi B}{1-y_2} + \frac{C}{1-y_2^2}\left[\lim_{\tau\to+\infty} \ln \frac{(1+y_2)\tau^2 - 2x_2\tau + 1 - y_2}{(1+w_2)\tau^2 + 2x_2\tau + 1 - y_2}\right] \\ & + \left[\frac{2D}{1-y_2} + \frac{2Cx_2}{1-y_2^2}\right] \frac{1+y_2}{x_1^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{\left(\frac{(1+y_2)\tau - x_2}{x_1}\right)^2 + 1}.\end{aligned}$$

Consequently, we obtain

$$\int_\gamma \langle \gamma(t), \nu(t)\rangle \mathrm{d}s = \frac{2\pi}{1-y_2}\left[B + \frac{1}{2}\left(D + \frac{Cx_2}{1+y_2}\right)\right].$$

An easy computation allows one to obtain the following values for constants,

$$B = \frac{x_1u_1}{1-x_1^2}, \qquad C = \frac{x_1v_1(1+y_2)}{1-x_1^2}, \qquad D = \frac{-x_1(u_1 + v_1x_2)}{1-x_1^2},$$

which gives

$$\int_\gamma k(s)\,\mathrm{d}s = -\int_{-\pi}^{\pi} \langle \gamma(t), \nu(t)\rangle \mathrm{d}t = 2\pi \frac{u_1}{(1-y_2)(1+x_1)}. \tag{6}$$

This formula has been calculated for the case $x_1 < 1$, but it can be easily seen that it works for the case $x_1 = 1$ too. The right hand side of (6) can be transformed in terms of $p = (x, 0, z)$ and $\nu(0) = (\nu_1, \nu_2, \nu_3)$ using that

$$x_1 = \frac{1}{r}, \qquad y_2 = \frac{\|p\|^2 - 1}{\|p\|^2 + 1}, \qquad u_1 = \frac{2\nu_1(\|p\|^2 + 1) - 4x\langle p, \nu(0)\rangle}{(\|p\|^2 + 1)^2},$$

to obtain

$$\int_\gamma k(s)\,\mathrm{d}s = -\frac{2\pi}{1+r} \langle p - \tilde{p}, \nu(0)\rangle, \tag{7}$$

where $\tilde{p} = (r, 0, 0)$.

All these computations imply important consequences. The first one allows one to control the total curvature of a Villarceau circle in the stereographic projection of a Hopf tube.

Theorem 8. *Let $\mathbf{N} = E_o(\mathcal{T})$ be the stereographic projection of a Hopf tube and $\gamma : [-\pi, \pi] \to \mathbf{N}$ a Villarceau circle with radius r which is determined by the point $p = \gamma(0) = (x, 0, z)$. Then, the total curvature of this curve in* $\mathbf{N}$

satisfies

$$\int_\gamma k(s)\,\mathrm{d}s \in \left[-2\pi\sqrt{\frac{r-1}{r+1}}, 2\pi\sqrt{\frac{r-1}{r+1}}\right].$$

Furthermore, the total curvature of γ attains its maximum and minimum if, and only if, the Gauss map of **N** *along γ is respectively*

$$N_0(\gamma(t)) = \mp\frac{\mathrm{d}\psi_t\left(\gamma'(0)\wedge\frac{p-\tilde{p}}{\|p-\tilde{p}\|}\right)}{\left\|\mathrm{d}\psi_t\left(\gamma'(0)\wedge\frac{p-\tilde{p}}{\|p-\tilde{p}\|}\right)\right\|}, \tag{8}$$

where $\tilde{p} = (r, 0, 0)$.

Proof. Just combine (7) with the following facts: First if the trace of γ is $\mathbb{S}^1 \times \{0\}$, that is $E_o(\mathbf{C}^\perp)$, then the total curvature vanishes identically no matter what the Gauss map of **N** is, and so the result holds trivially. Otherwise, notice that $\langle\gamma'(0), p-\tilde{p}\rangle = 0$. $\square$

Remark 9. The first part of this theorem remains valid when $\mathcal{T}$ is a lift of a curve in $\mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$ by the map Π_-. In this case the total curvature of γ attains its maximum and minimum if, and only if, the Gauss maps of **N** along γ are respectively

$$N_0(\gamma(t)) = \mp\frac{\mathrm{d}\chi_t\left(\gamma'(0)\wedge\frac{p-\tilde{p}}{\|p-\tilde{p}\|}\right)}{\left\|\mathrm{d}\chi_t\left(\gamma'(0)\wedge\frac{p-\tilde{p}}{\|p-\tilde{p}\|}\right)\right\|}, \tag{9}$$

where $\tilde{p} = (r, 0, 0)$.

Remark 10. Given a Villarceau circle γ in $\mathbb{R}^3$ of radius r, and given any value $\varrho \in \left[-2\pi\sqrt{\frac{r-1}{r+1}}, 2\pi\sqrt{\frac{r-1}{r+1}}\right]$, there exists a unit vector field N_o along γ and a soliton M with boundary conditions including (γ, N_o), such that the total curvature of γ in M takes the value $\int_\gamma \kappa = \varrho$.

As a second consequence, the topological charge carried by a soliton can be computed. Indeed, we have

Theorem 11. *Let $\phi \in I_\Gamma(M, \mathbb{R}^3)$ be a soliton of* **NSM**$_2$ *with boundary conditions $(\Gamma = \{\gamma_1, \gamma_2\}, N_0)$. Then it carries a topological charge given by*

$$Q(\phi) = 2\pi\sum_{i=1}^{2}\frac{\langle p_i - \tilde{p}_i, N_0 \wedge \gamma_i'(0)\rangle}{1+r_i},$$

where r_i is the radius of γ_i and $\tilde{p}_i = (r_i, 0, 0)$.

Certainly, we can fix a pair of Villarceau circles $\Gamma = \{\gamma_1, \gamma_2\}$ and then move the Gauss map, N_0, of solitons along the boundary Γ, to get, according to Theorem 8 and Remark 9, the maximum and the minimum topological charge.

7. Conclusions

We have developed a geometric algorithm for obtaining the whole moduli space of solitons in the **NSM**$_2$ that are foliated by Villarceau circles. In particular, this provides the first known examples of solitons in the **NSM**$_2$ that are foliated by nonparallel circles, in contrast with the Plateau integrable system case where such solutions do not exist. The criterion reduces the search for those solitons to that for clamped elastic curves in the once-punctured two-sphere of radius $\frac{1}{2}$.

The main ingredients involved in the method are: the principle of symmetric criticality, the Gauss–Bonnet formula, the extrinsic conformal invariance of the **NSM**$_2$ and the theory of clamped elasticae.

The algorithm works as follows:

(1) Let $\omega : I \subset \mathbb{R} \to \mathbb{R}$ be defined by

$$\omega(s) = \sqrt{2(\lambda^2 - 2)}\,\mathrm{cn}\left(\lambda s, \frac{\sqrt{\lambda^2 - 2}}{\sqrt{2}\lambda}\right),$$

where λ is a constant such that $\lambda \geq \sqrt{2}$ and $\mathrm{cn}(s, \rho)$ is the elliptic cosine of Jacobi.

(2) In the once-punctured two-sphere $\mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$, we choose an arc length curve $\alpha : I \subset \mathbb{R} \to \mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$ with curvature function ω. For any $[s_1, s_2] \subset I$, we put $\alpha(s_i) = m_i$, $\alpha'(s_i) = \vec{u}_i$ and $\vec{w}_i$ to denote the unit normal in m_i, $1 \leq i \leq 2$. Now, let $M_{s_1}^{s_2}(\alpha) = E_o(\Pi^{-1}(\alpha([s_1, s_2])))$, where E_o is the stereographic projection from $\zeta_o \in \mathbb{S}^3$ with $\Pi(\zeta_o) = m$.

(3) If $\mathbf{C}_i = \Pi^{-1}(m_i)$, then $M_{s_1}^{s_2}(\alpha)$ is a surface with boundary $\partial M_{s_1}^{s_2}(\alpha) = \Gamma = \{E_o(\mathbf{C}_1), E_o(\mathbf{C}_2)\}$ which is foliated by Villarceau circles.

(4) If W is the horizontal lift along $\{\mathbf{C}_1, \mathbf{C}_2\}$ such that $\mathrm{d}\Pi(W) = \vec{w}_i$, $1 \leq i \leq 2$, then

$$N_o = \frac{\mathrm{d}E_o(W)}{\|\mathrm{d}E_o(W)\|},$$

is a unit vector field along $\Gamma = \{E_o(\mathbf{C}_1), E_o(\mathbf{C}_2)\}$ which is $\mathbf{H}_C$-invariant. Hence, $M_{s_1}^{s_2}(\alpha)$ *is a soliton in the* $\mathbf{NSM}_2$ *with boundary conditions* (Γ, N_o).

Furthermore, we should point out that the very same construction works using Π_- instead of Π. Furthermore, all the solitons in the $\mathbf{NSM}_2$ that admit a Villarceau foliation are obtained this way.

The curvature function of a Villarceau circle in a soliton has been obtained. By evaluating the corresponding total curvature, we show that these solitons carry topological charges which can be holographically computed using the Gauss–Bonnet formula. Since those topological charges only depend on the boundary conditions, they are the same for fixed boundary conditions $(\Gamma = \{\gamma_1, \gamma_2\}, N_o)$.

Finally, using the terminology of the theory of submersions, the complete class of Villarceau foliated solitons in the $\mathbf{NSM}_2$ can be described as follows: First, consider the maps $\Phi, \Psi : \mathbb{R}^3 \to \mathbb{S}^2(\frac{1}{2}) \subset \mathbb{R}^3$, given by

$$\Phi(x, y, z) = \left(\frac{4xz + 2y(\Delta - 1)}{(\Delta + 1)^2}; \frac{4yz - 2x(\Delta - 1)}{(\Delta + 1)^2}; \frac{2\Delta - 4z^2 - \frac{1}{2}(\Delta - 1)^2}{(\Delta + 1)^2}\right),$$

$$\Psi(x, y, z) = \left(\frac{4xz - 2y(\Delta - 1)}{(\Delta + 1)^2}; \frac{4yz + 2x(\Delta - 1)}{(\Delta + 1)^2}; \frac{2\Delta - 4z^2 - \frac{1}{2}(\Delta - 1)^2}{(\Delta + 1)^2}\right),$$

where $\Delta = x^2 + y^2 + z^2$. Certainly, these are conformal Riemannian submersions with the usual metrics. However, if one removes the south pole, $m = (0, 0, -\frac{1}{2})$, of $\mathbb{S}^2(\frac{1}{2})$, then the above maps become principal circle fibre bundles with structure groups $\mathbf{H}_\mathbf{C}^+$ and $\mathbf{H}_\mathbf{C}^-$, respectively

$$\Phi, \Psi : \mathbb{R}^3 \setminus (z\text{-axis}) \to \mathbb{S}^2\left(\frac{1}{2}\right) \setminus \{m\}, \qquad \Phi = \Pi \circ E_o^{-1}, \qquad \Psi = \Pi_- \circ E_o^{-1},$$

and the fibres of both fibrations are the Villarceau circles of first and second-kind, respectively. Now, a surface in Euclidean three-space is Villarceau foliated if, up to motions, it is the complete lifting, via Φ or Ψ, of a curve in the two-sphere. In particular, since the $\mathbf{NSM}_2$ is invariant under conformal changes in the Euclidean metric, the principle of symmetric criticality allows one to get the Villarceau foliated solitons as complete liftings of clamped pieces of elasticae in $\mathbb{S}^2(\frac{1}{2}) \setminus \{m\}$.

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